

# HOMOCLINIC INTERSECTIONS OF SYMPLECTIC PARTIALLY HYPERBOLIC SYSTEMS WITH 2D CENTER

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**ABSTRACT.** We study some generic properties of partially hyperbolic symplectic systems with 2D center. We prove that  $C^r$  generically, every hyperbolic periodic point has a transverse homoclinic intersection for the maps close to a direct/skew product of an Anosov diffeomorphism with a map on  $S^2$  or  $\mathbb{T}^2$ .

## 1. INTRODUCTION

Let  $M$  be a closed manifold and  $f : M \rightarrow M$  be a diffeomorphism on  $M$ . A periodic point  $p = f^n p$  is said to be *hyperbolic*, if the linearization  $D_p f^n : T_p M \rightarrow T_p M$  doesn't admit any eigenvalue of norm 1. Associated to a hyperbolic periodic orbit are the *stable* and *unstable manifolds*  $W^{s,u}(p)$  of  $p$ . A point in the intersection  $W^s(p) \cap W^u(q)$  for another hyperbolic periodic point  $q$  is called a *heteroclinic intersection* (a *homoclinic intersection* if  $q = p$ ). Note that the intersection  $W^s(p) \cap W^u(q)$  may not be *transverse* (even when  $p$  and  $q$  have the same stable dimension).

Poincaré was the first one to consider the phase portrait when there exists a transverse homoclinic intersections during his study of the  $n$ -body problem around 1890. Later in [Poin], Poincaré described the phenomenon that any transverse homoclinic intersection is accumulated by many other homoclinic points and this mechanism generates various complicated dynamical behaviors. This mechanism was developed by Birkhoff, who showed in [Bir35] the persistent existence of *infinitely many* hyperbolic periodic points whenever there is a transverse homoclinic intersection, and by Smale, who introduced in [Sma65] the geometric model, now called *Smale horseshoe*, for the dynamics around a transverse homoclinic intersection, and started a systematic study of general hyperbolic sets. Melnikov developed in [Mel63] a method for detecting homoclinic intersections in dynamical systems (this has also been used by Poincaré). Poincaré conjectured in [Poin] that for a generic  $f \in \text{Diff}_\mu^r(M)$ , and for every hyperbolic periodic point  $p$  of  $f$ ,

- (P1) the set of (hyperbolic) periodic points is *dense* in the space  $M$ ;
- (P2a)  $W^s(p) \cap W^u(p) \setminus \{p\} \neq \emptyset$  (weak version);
- (P2b)  $W^s(p) \cap W^u(p)$  is *dense* in  $W^s(p) \cup W^u(p)$  (strong version).

These conjectures are closely related to the *Closing Lemma* and *Connecting Lemma*, see [Pug11] for a historic account of these terminologies. Poincaré's conjectures have been one of the main motivations for the recent development in dynamical systems. All three parts have been proved for  $r = 1$ : (P1) follows from Pugh's closing lemma [Pug67a, Pug67b, PuRo83], (P2a) was proved by Takens in [Tak72], (P2b) was proved by Xia [Xia96]. There are some special classes of maps that (P1)–(P2b) hold everywhere (not only generically): Anosov's *uniformly hyperbolic* systems [Ano67], and Pesin's *nonuniformly hyperbolic* systems [Pes77]. There are some partial results for (P1) and (P2a) for systems beyond hyperbolicity when  $r > 1$  (mainly in 2D). Robinson proved in [Rob73] that on two-sphere, if the unstable manifold of a hyperbolic fixed point accumulates on its stable manifold, then a  $C^r$  small perturbation can create a homoclinic intersection. Pixton [Pix82] extended Robinson's result to periodic orbits, and proved that (P2a) holds on  $S^2$ . That is, for a  $C^r$  generic area-preserving diffeomorphism on  $S^2$ , there exist some homoclinic orbits for every

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hyperbolic periodic point. Using some topological argument in [Oli87], Oliveira showed the generic existence of homoclinic orbits on  $\mathbb{T}^2$ . His result was extended in [Oli00] to any compact surface (among those whose induced actions  $f_*$  on the first homology group  $H_1(M)$  are irreducible). Then Xia proved in [Xia06b] the generic existence of homoclinic orbits on general compact surface among the class of systems homotopic to identity, and among the class of Hamiltonian diffeomorphisms. Recently in [XZ14, Zh15], we proved the existence of homoclinic intersections for every hyperbolic periodic point for generic convex billiards on  $\mathbb{R}^2$  and  $S^2$ , respectively.

In this paper we study the homoclinic intersections of some symplectic partially hyperbolic systems. As an illustration of our main result, let's start with a special case. Let  $(M, \omega)$  be a closed symplectic manifold,  $f \in \text{Diff}_\omega^r(M)$  be a symplectic Anosov diffeomorphism. Let  $S$  be a closed surface with an area-form  $\mu$  on  $S$ , and  $g \in \text{Diff}_\mu^r(S)$  such that  $f \times g$  is partially hyperbolic with center bundle  $E_{(x,s)}^c = \{0_x\} \times T_s S$ . Replacing  $f$  by  $f^n$  for large enough  $n$  if necessary, we may assume  $f \times g$  is  $r$ -normally partially hyperbolic. Let  $M' = M \times S$  and  $\omega' = \omega \oplus \mu$ . Then  $f \times g \in \text{PH}_{\omega'}^r(M')$ , and there exists a  $C^1$  open neighborhood  $\mathcal{U} \subset \text{PH}_{\omega'}^r(M')$  of  $f \times g$  such that each  $h \in \mathcal{U}$  is  $r$ -normally partially hyperbolic with stably integrable center bundle. Moreover, the foliation  $\mathcal{F}_h^c$  is leaf conjugate to the trivial foliation  $\{\{x\} \times S : x \in M\}$ . Therefore, each center leaf  $\mathcal{F}_h^c(x, s)$  is diffeomorphic to the surface  $S$ . This class of maps in  $\mathcal{U}$  have been studied in [XZ06], where they proved (P1). That is,  $C^r$ -generically in  $\mathcal{U}$ , the set of (hyperbolic) periodic points are dense in  $M$ . In this paper, we show

**Theorem 1.** *Let  $S$  be diffeomorphic to either the 2-sphere  $S^2$  or 2-torus  $\mathbb{T}^2$ . Then there is a small neighborhood  $\mathcal{U} \subset \text{PH}_{\omega'}^r(M')$  of  $f \times g$ , such that for a  $C^r$ -generic  $h \in \mathcal{U}$ ,  $W^s(p) \cap W^s(p) \setminus \{p\} \neq \emptyset$  for each hyperbolic periodic point  $p$  of  $h$ .*

More generally, let's consider a skew product system. That is, let  $f \in \text{Diff}_\omega^r(M)$  be a symplectic Anosov diffeomorphism, and  $\phi : M \rightarrow \text{Diff}_\mu^r(S)$  be a  $C^r$  smooth cocycle over  $M$ . Let  $F : M \times S \rightarrow M \times S$ ,  $(x, s) \mapsto (fx, \phi(x)(s))$  be the induced skew product of  $\phi$  over  $f$ . Then the subbundle  $E \subset TM'$  with  $E_{(x,s)}^c = \{0_x\} \times T_s S$  is  $DF$ -invariant. Replacing  $f$  by  $f^n$  for large enough  $n$  if necessary, we may assume  $F$  is  $r$ -normally partially hyperbolic. Similarly, we show that

**Theorem 2.** *Let  $S$  be diffeomorphic to either  $S^2$  or  $\mathbb{T}^2$ . Let  $f$  and  $\phi$  be given as above, and  $F$  be the skew product of  $\phi$  over  $f$ . Then there is a small neighborhood  $\mathcal{U} \subset \text{PH}_{\omega'}^r(M')$  of  $F$ , such that for a  $C^r$ -generic  $h \in \mathcal{U}$ ,  $W^s(p) \cap W^s(p) \setminus \{p\} \neq \emptyset$  for each hyperbolic periodic point  $p$  of  $h$ .*

Now let's state our main result. Let  $(M, \omega)$  be a closed symplectic manifold,  $\text{PH}_\omega^r(M)$  be the set of partially hyperbolic symplectic diffeomorphisms on  $M$ , and  $\mathcal{N}^r \subset \text{PH}_\omega^r(M)$  be the set of partially hyperbolic maps such that the partially hyperbolic splitting is  $r$ -normally hyperbolic and the center bundle is stably integrable. Let  $\text{PH}_\omega^r(M, 2)$  be those map in  $\text{PH}_\omega^r(M)$  with 2-dimensional center, and  $\mathcal{N}^r(2) = \mathcal{N}^r \cap \text{PH}_\omega^r(M, 2)$  be the set of partially hyperbolic maps in  $\mathcal{N}^r$  with 2D center bundles.

**Theorem 3.** *There is a residual subset  $\mathcal{R} \subset \mathcal{N}^r(2)$  such that for each  $f \in \mathcal{R}$ , for each hyperbolic periodic point  $p$  of  $f$ , if  $\mathcal{F}_f^c(p)$  is homeomorphic to either  $S^2$  or  $\mathbb{T}^2$ , then  $W^s(p) \cap W^s(p) \setminus \{p\} \neq \emptyset$ .*

Note that the center foliation  $\mathcal{F}_f^c$  may not be (uniformly) compact, even there are some compact leaves. An intuitive example is the orbit foliation of a Anosov flow: a leaf is compact if and only if it is a periodic orbit. It is clear that Theorem 1 and 2 follow directly from the above theorem.

Now let's sketch the proof of Theorem 3. We start with a result on Kupka–Smale properties of generic symplectic systems proved by Robinson, then make a local perturbation on the center leaf around each nonhyperbolic periodic point such that the restricted dynamics in the corresponding center leaf is Moser stable. Next we use the generating function to lift the local center-leaf perturbation to a local perturbation of the ambient manifold. Then we apply the prime-end theory developed by Mather to deduce the recurrence of all stable and unstable manifolds restricted on the center leaf. Simple topology assumption, that is, the center leaf is homeomorphic to either  $S^2$  or  $\mathbb{T}^2$ , ensures the existence of homoclinic intersections for all hyperbolic periodic points.

## 2. PRELIMINARIES

Let  $M$  be a closed manifold endowed with some Riemannian metric,  $\text{Diff}^r(M)$  be the set of  $C^r$  diffeomorphisms on  $M$ . Let  $TM = E \oplus F$  be a splitting of  $TM$  into two  $Df$ -invariant subbundles. Then we say that  $E$  is *dominated* by  $F$ , if there exists  $n \geq 1$  such that for any  $x \in M$ ,

- $2\|D_x f^n(u)\| < \|D_x f^n(v)\|$  for any unit vectors  $u \in E_x$  and  $v \in F_x$ .

Note that both  $E$  and  $F$  are continuous subbundles of  $TM$ . Then  $f$  is said to be *partially hyperbolic*, if there exist a three-way splitting  $TM = E^s \oplus E^c \oplus E^u$ , such that

- (1)  $E^s$  is dominated by  $E^c \oplus E^u$ , and  $E^s \oplus E^c$  is dominated by  $E^u$ ;
- (2) there exists  $k \geq 1$  such that  $2\|D_x f^k|_{E_x^s}\| \leq 1$  and  $2\|D_x f^{-k}|_{E_x^u}\| \leq 1$ .

In particular,  $f$  is said to be *Anosov* (or equivalently, uniformly hyperbolic), if  $E^c = \{0\}$ .

The above definition of partially hyperbolic maps is elegant. In the following we give an equivalent, but more easy-to-use definition for later convenience. Recall that a function  $\phi : M \rightarrow \mathbb{R}$  induces a *multiplicative cocycle*  $\{\phi_n : n \geq 0\}$  on  $M$ , where  $\phi_0(x) \equiv 1$ , and  $\phi_n(x) = \prod_{k=0}^{n-1} \phi(f^k x)$  for each  $x \in M$  and for all  $n \geq 1$ .

**Definition 2.1.** The map  $f$  is said to be *partially hyperbolic*, if there exist a three-way splitting  $TM = E^s \oplus E^c \oplus E^u$ , a constant  $C \geq 1$ , four continuous functions  $\nu, \hat{\nu}, \gamma$  and  $\hat{\gamma} : M \rightarrow \mathbb{R}$  with  $\nu, \hat{\nu} < 1$  and  $\nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$ , such that for any  $x \in M$ , and for any unit vector  $v \in T_x M$ ,

$$\|Df^n(v)\| < C \cdot \nu_n(x), \quad \text{if } v \in E_x^s, \quad (2.1)$$

$$C^{-1} \cdot \gamma_n(x) < \|Df^n(v)\| < C \cdot \hat{\gamma}_n(x)^{-1}, \quad \text{if } v \in E_x^c, \quad (2.2)$$

$$C^{-1} \cdot \hat{\nu}_n(x)^{-1} < \|Df^n(v)\|, \quad \text{if } v \in E_x^u. \quad (2.3)$$

Generally speaking, the partially hyperbolic splitting of  $f$  may not be unique. Note that the stable bundle  $E^s$  is uniquely integrable. Let  $\mathcal{F}_f^s$  be the stable foliation of  $f$ , whose leaves  $\mathcal{F}_f^s(x)$  are  $C^r$  immersed submanifolds. So is the unstable one, and denote the unstable foliation by  $\mathcal{F}_f^u$ . However, the center bundle  $E^c$  may *not* be integrable, and when integrable, the center leaves may not be  $C^r$ .

**2.1. Dynamical coherence.** A partially hyperbolic map  $f$  is said to be *dynamically coherent*, if there exists an  $f$ -invariant foliation  $\mathcal{F}_f^c$  such that  $T_x \mathcal{F}_f^c(x) = E_x^c$  for every  $x \in M$ . Then  $f$  is said to be *stably dynamically coherent*, if there is an open neighborhood  $\mathcal{U} \subset \text{PH}^r(M)$ , such that every  $g \in \mathcal{U}$  is dynamically coherent.

Note that there are several versions of definitions of dynamical coherence in the literature. See [BW08] for more details.

**Proposition 2.1** ([HPS]). *Let  $f \in \text{PH}^r(M)$  such that the center foliation  $\mathcal{F}^c$  is  $C^1$ . Then  $f$  is stably dynamically coherent.*

As a direct corollary, a product system  $f_1 \times f_2 \in \text{PH}^r(M_1 \times M_2)$  is stably dynamically coherent.

Plaque expansiveness, a condition weaker than  $\mathcal{F}^c$  being  $C^1$ , is introduced in [HPS]. They showed that if  $\mathcal{F}^c$  is plaque expansive, then  $f$  is stably dynamically coherent.

**2.2. Normal hyperbolicity.** Let  $f \in \text{PH}_\omega^r(M)$ , and  $\nu, \gamma, \hat{\nu}$  and  $\hat{\gamma}$  be the functions given in Definition 2.1. Then  $f$  is said to be  *$r$ -normally hyperbolic*, if  $\nu < \gamma^r$  and  $\hat{\nu} < \hat{\gamma}^r$ . It follows from the definition that every partially hyperbolic diffeomorphism is  $r$ -normally hyperbolic, for some  $r \geq 1$ .

**Proposition 2.2** ([HPS]). *Let  $f \in \text{PH}_\omega^r(M)$  such that  $E^c$  is integrable. If  $f$  is  $r$ -normally hyperbolic, then all center leaves of  $\mathcal{F}^c$  are  $C^r$ .*

**2.3. Symplectic systems.** A  $2n$ -dimensional manifold  $M$  is said to be *symplectic*, if there exists a closed nondegenerate 2-form  $\omega$  on  $M$ . Let  $E \subset TM$  be a continuous subbundle such that  $\dim(E_x) = i$  for any  $x \in M$ . In this case we also denote it by  $\dim E = i$ . The *symplectic orthogonal complement* of  $E$ , denoted by  $E^\omega$ , is given by  $E_x^\omega = \{v \in T_x M : \omega(v, w) = 0 \text{ for any } w \in E_x\}$ . Clearly  $\dim E^\omega = 2n - i$ . A subbundle  $E$  is said to be *isotropic*, if  $E \subset E^\omega$ ; is said to be *coisotropic*, if  $E \supset E^\omega$ ; is said to be *symplectic*, if  $E \cap E^\omega = 0$ ; and is said to be *Lagrangian*, if  $E = E^\omega$ .

Let  $\text{Diff}_\omega^r(M)$  be the set of symplectic diffeomorphisms  $f : M \rightarrow M$ , that is,  $f^*\omega = \omega$ . Similarly, let  $\text{PH}_\omega^r(M)$  be the set of symplectic partially hyperbolic diffeomorphisms on  $M$ . Note that for a given map  $f \in \text{PH}_\omega^r(M)$ , the partially hyperbolic splitting of  $f$  may not be unique. However, the center bundle can always be chosen to be a symplectic subbundle of  $TM$ .

**Proposition 2.3** ([SX06]). *Let  $f \in \text{Diff}_\omega^r(M)$ , and  $TM = E \oplus F$  be a  $Df$ -invariant splitting of  $f$  with  $\dim E < \dim F$  such that  $E$  is dominated  $F$ . Then  $f$  is partially hyperbolic, where  $E^s = E$ ,  $E^c = E^\omega \cap F$  and  $E^u = (E^c)^\omega \cap F$ . Moreover,  $E^s$  and  $E^u$  are isotropic,  $E^s \oplus E^u$  and  $E^c$  are symplectic and are skew-orthogonal to each other.*

In the following the partially hyperbolic splitting  $TM = E^s \oplus E^c \oplus E^u$  for  $f \in \text{PH}_\omega^r(M)$  will be fixed such that the center bundle  $E^c$  and the combined bundle  $E^s \oplus E^u$  are symplectic. In particular, we have

**Corollary 1** ([XZ06]). *Let  $f \in \text{PH}_\omega^r(M)$ , and  $\mathcal{F}^c$  is an  $f$ -invariant foliation tangent to the center bundle  $E^c$  of  $f$ . Then the center leaves  $\mathcal{F}^c(x)$  are symplectic (possibly immersed) submanifolds with respect to the restricted symplectic form  $\omega|_{\mathcal{F}_f^c(x)}$ . Moreover, the restrictions of  $f$  from  $\mathcal{F}_f^c(x) \rightarrow \mathcal{F}_f^c(fx)$  are symplectic diffeomorphisms.*

Moreover, it is proved in [SX06] that symplectic partially hyperbolic maps are symmetric. That is, one can take  $\hat{\nu} = \nu$  and  $\hat{\gamma} = \gamma$  in Definition 2.1.

**Proposition 2.4.** *If  $f \in \text{PH}_\omega^r(M)$ , then there exist a constant  $C \geq 1$ , two continuous functions  $\gamma, \nu : M \rightarrow (0, 1)$  such that for any  $x \in M$ , and for any unit vector  $v \in T_x M$ ,*

$$\|Df^n(v)\| < C \cdot \nu_n(x), \quad \text{if } v \in E_x^s, \quad (2.4)$$

$$C^{-1} \cdot \gamma_n(x) < \|Df^n(v)\| < C \cdot \gamma_n(x)^{-1}, \quad \text{if } v \in E_x^c, \quad (2.5)$$

$$C^{-1} \cdot \nu_n(x)^{-1} < \|Df^n(v)\|, \quad \text{if } v \in E_x^u. \quad (2.6)$$

**Remark 2.1.** The normal hyperbolicity condition defined in §2.2 for general partially hyperbolic maps admits a simpler form in the symplectic case. That is, a map  $f \in \text{PH}_\omega^r(M)$  is said to be  $r$ -normally hyperbolic, if the two functions  $\nu$  and  $\gamma$  in Proposition 2.4 satisfy  $\nu < \gamma^r$ .

**2.4. Surface maps.** Let  $S$  be a 2D surface, and  $g : S \rightarrow S$  be a  $C^r$  symplectic map fixing a point  $p \in S$ . Let  $\lambda_p$  and  $\lambda_p^{-1}$  be the two eigenvalues of  $D_p g : T_p S \rightarrow T_p S$ . Then  $p$  is said to be *hyperbolic* if  $|\lambda_p| \neq 1$ , be *parabolic* if  $\lambda_p = \pm 1$ , and be *elliptic* if otherwise.

**Proposition 2.5.** *Suppose  $p$  is an elliptic fixed point of  $g$  such that  $\lambda_p^j \neq 1$  for each  $1 \leq j \leq 4$ . Then there exists a real-analytic symplectic diffeomorphism  $h$ , defined on a neighborhood of 0 in  $\mathbb{C}$  with  $h(0) = p$ , such that in the complex coordinate  $z = x + iy$ , one has:*

$$h^{-1} \circ g \circ h(z) = \lambda_p \cdot z \cdot e^{2\pi i a_1 |z|^2} + o(|z|^3), \quad (2.7)$$

where  $a_1 = a_1(g)$  depends continuously on  $g$ .

For a proof of above theorem, see [Mos73]. The formulation on the right side of (2.7) is called the *Birkhoff normal form* of  $g$  at  $p$ , and  $a_1 = a_1(p; g)$  is called the (first) *Birkhoff coefficient* of  $g$  at  $p$ .

**Definition 2.2.** An elliptic fixed point  $p$  of a surface map  $g : S \rightarrow S$  is said to be *Moser stable*, if there is a fundamental system  $\{D_n\}$  of nesting neighborhoods in  $S$  around  $p$ , where each  $D_n$  is an invariant closed disk surrounding the point  $p$ , such that the restriction of  $g$  on  $\partial D_n \simeq S^1$  is transitive (minimal).

Note that Moser stable periodic points are isolated from the dynamics in the sense that it can not be reached from any invariant ray whose starting point lies outside some  $D_n$ .

The following is Moser's *Twisting Mapping Theorem* (see [Mos73]):

**Theorem 4.** *Let  $p$  be an elliptic fixed point of  $g$  and  $a_1 = a_1(p; g)$  be the Birkhoff coefficient of  $g$  at  $p$ . If  $a_1 \neq 0$ , then  $p$  is Moser stable.*

**2.5. Robinson's result on Kupka–Smale property.** Let  $f \in \text{Diff}_\omega^r(M)$ ,  $n \geq 1$  and  $P_n(f)$  be the set of points fixed by  $f^n$ . Clearly  $P_n(f)$  is a closed set. Let  $p$  be a periodic point of  $f$  of period  $k$ . Then  $p$  is said to be *hyperbolic* if  $|\lambda| \neq 1$  for any eigenvalue of the linearization  $D_p f^k : T_p M \rightarrow T_p M$  of  $f$  at  $p$ . Given a hyperbolic periodic point  $p$  of  $f$ , let  $W^s(p)$  and  $W^u(p)$  be the stable and unstable manifolds of  $p$ .

More generally, a periodic point  $p$  of minimal period  $n$  is said to be  *$N$ -elementary*, if  $\lambda^k \neq 1$  for each  $1 \leq k \leq N$  and for each eigenvalue  $\lambda$  of  $D_p f^n : T_p M \rightarrow T_p M$ . Then  $p$  is said to be *elementary*, if it is  $N$ -elementary for any  $N \geq 1$ . Robinson proved in [Rob70] the following property:

**Proposition 2.6.** *There exists an open and dense subset  $\mathcal{U}_n^r \subset \text{Diff}_\omega^r(M)$ , such that for each  $f \in \mathcal{U}_n^r$ ,*

- (1)  $P_n(f)$  is finite and depends continuously on  $f$ , and each periodic point in  $P_n(f)$  is  $n$ -elementary;
- (2) for any two  $p, q \in P_n(f)$ ,  $W_f^u(p, n) \pitchfork W_f^s(q, n)$ .

**Remark 2.2.** Let  $\mathcal{R}_{KS} = \bigcap_{n \geq 1} \mathcal{U}_n^r$ : then  $\mathcal{R}_{KS}$  contains a  $C^r$ -residual subset of  $\text{Diff}_\omega^r(M)$ , and each  $f \in \mathcal{R}_{KS}$  is Kupka–Smale. That is,

- (1) each periodic point of  $f$  is elementary;
- (2)  $W_f^u(p) \pitchfork W_f^s(q)$  for any two hyperbolic periodic points  $p, q$ .

The second item of the above property says that, when  $W^s(p)$  and  $W^u(q)$  have a nontrivial intersection, the intersection is actually transverse. However, it does not address the question whether  $W^s(p)$  and  $W^u(q)$  can have any nontrivial intersection. Theorem 3 confirms the existence of homoclinic intersections of every hyperbolic periodic point generically.

**Remark 2.3.** It is proved in [Rob73, Pix82] on  $S^2$ , and [Oli87] on  $\mathbb{T}^2$  that  $C^r$  generically, every hyperbolic periodic point admits transverse homoclinic points. For the maps  $f \in \mathcal{R}_{KS}$ , the center-leaf maps  $f^k : \mathcal{F}_p^c \rightarrow \mathcal{F}_p^c$  (counting to periods) may or may not be the generic ones. It is not clear if one can tell whether such a given center-leaf diffeomorphism satisfies their genericity requirement. So we need to a *handy* criterion for proving existence of homoclinic points for the center-leaf maps, see §5.

### 3. SOME PERTURBATION RESULTS

In this section we will give some perturbation results about partially hyperbolic symplectic diffeomorphisms with 2D center. Let  $\text{PH}_\omega^r(M, 2)$  be the set of partially hyperbolic maps  $f \in \text{PH}_\omega^r(M)$  with center dimension  $\dim E^c = 2$ . Let  $\mathcal{N}^r(2)$  be the set of partially hyperbolic maps  $f \in \text{PH}_\omega^r(M, 2)$  that are  $r$ -normally hyperbolic whose center bundles are stably integrable. It is evident that  $\mathcal{N}^r(2)$  is an open subset of  $\text{PH}_\omega^r(M, 2)$ .

Let  $p$  be a periodic point of  $f$  of period  $n$ . Then the splitting  $T_p M = E_p^s \oplus E_p^c \oplus E_p^u$  are  $D_p f^n$ -invariant, and the eigenvalues of  $D_p f^n$  along the two hyperbolic directions have modulus different from 1. The two eigenvalues of  $D_p f^n$  along the center direction  $E_p^c$  satisfy  $\lambda_1^c \cdot \lambda_2^c = 1$ . Therefore,

- either  $|\lambda_i^c| \neq 1$  for both  $i = 1, 2$ . In this case  $p$  is a hyperbolic periodic point of  $f$ ;
- or  $|\lambda_i^c| = 1$  for both  $i = 1, 2$ . In this case  $p$  is nonhyperbolic with a 2D neutral direction.

**Remark 3.1.** Note that for  $f \in \text{PH}_\omega^r(M)$ , for any *hyperbolic* periodic point  $p$ , any eigenvalue  $\lambda$  in with eigenvector  $v \in E_p^s \oplus E_p^u$  has norm different from 1, and  $\mathcal{F}^{s,u}(p) \subset W^{s,u}(p)$ , respectively. However,  $\mathcal{F}^{s,u}(p)$  is *always* strictly contained in  $W^{s,u}(p)$ , since there are contraction/expansion along the center direction  $E_p^c$ .

**Proposition 3.1.** *There exists an open and dense subset  $\mathcal{V}_n \subset \mathcal{N}^r(2)$  such that for each  $f \in \mathcal{V}_n$  and each periodic point  $p \in P_n(f)$ , either  $p$  is hyperbolic, or the center-leaf Birkhoff coefficient  $a_1(p, f^k, \mathcal{F}_f^c(p)) \neq 0$ , where  $k$  is the minimal period of  $p$ .*

*Proof.* Let  $\mathcal{U}_n^r(2) = \mathcal{N}^r(2) \cap \mathcal{U}_n^r$ , where  $\mathcal{U}_n^r$  is the open and dense subset given in Proposition 2.6. Let  $f \in \mathcal{U}_n^r(2)$ , and  $p \in P_n(f)$  be a point fixed by  $f^n$ , and  $k$  be the minimal period of  $p$ . Then  $k|n$ . In this case, the center leaf  $\mathcal{F}_f^c(p)$  of  $p$  is also invariant under  $f^k$ , and we can consider the restriction of  $f^k$  on  $\mathcal{F}_f^c(p)$ , which is a symplectic surface diffeomorphism.

Since  $f \in \mathcal{U}_n^r$ , we see that  $p$  is 4-elementary, and hence the center eigenvalue  $\lambda_1^j \neq 1$  for each  $1 \leq j \leq 4$ . Then we can define the Birkhoff normal form around  $p$  in the center leaf  $\mathcal{F}_f^c(p)$ , and let  $a_1(p, f^k, \mathcal{F}_f^c(p))$  be the first Birkhoff coefficient of this center-leaf Birkhoff normal form at  $p$ . Let  $\mathcal{U}$  be an open neighborhood of  $f$  in  $\mathcal{U}_n^r$  such that  $P_n(\cdot)$  has constant cardinality and varies continuously on  $\mathcal{U}$ .

**Claim.** If  $a_1(p, f^k, \mathcal{F}_f^c(p)) \neq 0$ , then there exists an open neighborhood  $\mathcal{U}(f, p) \subset \mathcal{U}$  of  $f$  such that  $a_1(p_g, g^k, \mathcal{F}_g^c(p_g)) \neq 0$  for all  $g \in \mathcal{U}(f, p)$ .

*Proof of Claim.* Firstly note that  $p$  is nondegenerate. Let  $p_g$  be the continuation of  $p$  for maps  $g$  close to  $f$ . Moreover, the partially hyperbolic splitting on the maps  $g$  depends continuously on  $g$ , and  $g$  admits a  $g$ -invariant center foliation  $\mathcal{F}_g^c$ . Therefore, the system  $g \mapsto (g^k, \mathcal{F}_g^c(p_g))$  varies continuously, so is the Birkhoff coefficient  $g \mapsto a_1(p_g, g^k, \mathcal{F}_g^c(p_g))$ . This completes the proof of Claim.  $\square$

In the following we assume  $a_1(p, f^k, \mathcal{F}_f^c(p)) = 0$ . Then we make a  $C^r$ -small local perturbation on the center leaf, say  $h_c : \mathcal{F}_f^c(p) \rightarrow \mathcal{F}_f^c(p)$ , supported on a small neighborhood  $U_c \subset \mathcal{F}_f^c(p)$  of  $p$ , such that  $h_c(p) = p$ ,  $h_c(\mathcal{F}_f^c(p)) = \mathcal{F}_f^c(p)$  and the Birkhoff coefficient  $a_1(p; f^k \circ h_c, \mathcal{F}_f^c(p)) \neq 0$ . Note that  $k$  is the period of  $p$ , not the center leaf  $\mathcal{F}_f^c(p)$ . In particular it is possible that  $f^j \mathcal{F}_f^c(p) = \mathcal{F}_f^c(p)$  for some  $j|k$ . In this case the intersection  $\mathcal{O}(p, f) \cap \mathcal{F}_f^c(p)$  is a finite set, and the support of  $h_c$  can be made small enough such that it does not interfere with the intermediate returns of  $p$  to  $\mathcal{F}_f^c(p)$ . However, note that the map  $h_c$  hasn't been defined on  $M \setminus \mathcal{F}_f^c(p)$ . Next we will extend  $h_c$  to the whole manifold  $M$ .

By Darboux's theorem, there exists a local coordinate system  $(x, y)$  on  $U_c$  around  $p$  such that the restriction  $\omega_{U_c} = dx \wedge dy$ . Let  $h_c(x, y) = (X, Y)$ . Then  $YdX - ydx$  is a *close 1-form* on  $U_c$  and hence also *exact* (since  $U_c$  is simply connected). So  $YdX - ydx = dS_c$  for some function  $S_c$  which is identically 0 on  $F_f^c(p) \setminus U_c$ . Note that  $S_c$  is  $C^{r+1}$ -small, and is called a *generating function* of  $h_c$ .

Note that all iterates  $F_f^c(f^k p)$  are compact leaves. Using Darboux's theorem again, one can extend the local coordinate system  $(x, y)$  on  $U_c \subset F_f^c(p)$  to a local neighborhood  $U \subset M$  containing  $U_c$ , say  $(x, y, x'_i, y'_i)$ , such that  $\omega = dx \wedge dy + \sum_i dx'_i \wedge dy'_i$ . Then we extend the above *center-leaf* generating function  $S_c$  to a generating function  $S$  supported on  $U$  such that  $S|_{U_c} = S_c$ . Let  $h$  be the corresponding symplectic diffeomorphism generated by  $S$ . Note that  $h = Id$  on  $M \setminus U$ ,  $h$  is  $C^r$ -close to identity and  $h|_{F_f^c(p)} = h_c$ . Let  $g = f \circ h$ . Then we have  $g^i(p) = f^i \circ h(p) = f^i(p)$  for each  $1 \leq i \leq k$ ,  $g^k(F_f^c(p)) = F_f^c(p)$  and  $a_1(p, g^k, F_g^c(p)) = a_1(p, f^k \circ h_c, \mathcal{F}_f^c(p)) \neq 0$ . Note that any invariant  $r$ -normally hyperbolic manifold is isolated and persists under perturbations. Then the fact  $\mathcal{F}_f^c(p)$  is an  $r$ -normally hyperbolic manifold of  $g^k$  implies that  $\mathcal{F}_g^c(p) = \mathcal{F}_f^c(p)$ . Therefore, we can rewrite the above conclusion as  $a_1(p, g^k, F_g^c(p)) \neq 0$ .

Applying the previous claim again, we have that there is an open neighborhood  $\mathcal{U}(p, g) \subset \mathcal{U}$  of  $g$  such that for any  $h \in \mathcal{U}(p, g)$ , the continuation  $p_h$  satisfies  $a_1(p_h, h^k, F_h^c(p_h)) \neq 0$ . Let  $\tau = |P_n(g)|$ , which is a constant on  $\mathcal{U}$ . Then by induction, we can find an open subset  $\mathcal{U}^{(\tau)} \subset \mathcal{U}(p, g)$ , such that for each  $g \in \mathcal{U}^{(\tau)}$  and each periodic point  $p_g \in P_n(g)$ , either  $p_g$  is hyperbolic, or the center-leaf Birkhoff coefficient  $a_1(p_g, g^k, \mathcal{F}_g^c(p_g)) \neq 0$ , where  $k$  is the minimal period of  $p_g$ .

Note that our  $f$  is chosen arbitrarily in  $\mathcal{U}_n^r(2)$ , and  $\mathcal{U}^{(\tau)}$  contains an open set in an arbitrarily small open neighborhood  $\mathcal{U}$  of  $f$ . Putting these sets  $\mathcal{U}^{(\tau)}$  together, we get an open and dense subset in  $\mathcal{U}_n^r(2)$ , say  $\mathcal{V}_n$ , such that for each  $f \in \mathcal{V}_n$  and each periodic point  $p \in P_n(f)$ , either  $p$  is hyperbolic,

or the center-leaf Birkhoff coefficient  $a_1(p, f^k, F_f^c) \neq 0$ , where  $k$  is the minimal period of  $p$ . Then it follows that  $\mathcal{V}_n$  is an open and dense subset of  $\mathcal{N}^r$ .  $\square$

**Remark 3.2.** The perturbation  $h$  constructed in the proof is localized around  $p$ , and does not interfere with the dynamics around the other iterates  $\mathcal{F}_f^c(f^k p)$ . Therefore,  $\mathcal{F}_g^c(g^i p) = \mathcal{F}_f^c(f^i p)$  for all  $0 \leq i < k$ . However, the partially hyperbolic splitting of  $g$  and the center foliation  $\mathcal{F}_g^c$  are not the same after the perturbation  $f$ . In particular, most of the center leaves  $\mathcal{F}_g^c(x)$  are slightly deformed comparing to  $\mathcal{F}_f^c(x)$ .

Let  $\mathcal{V}_n$  be the open set given in Proposition 3.1, and  $\mathcal{R} = \bigcap_n \mathcal{V}_n$ . Then  $\mathcal{R}$  contains a residual subset of  $\mathcal{N}^r(2)$ .

**Proposition 3.2.** *Let  $f \in \mathcal{R}$ . We have that*

- (1)  $P_n(f)$  is finite, and each periodic point is elementary;
- (2)  $W^s(p) \pitchfork W^u(q)$  for any two hyperbolic periodic points  $p, q$ ;
- (3) the center Birkhoff coefficient  $a_1(p, f^k, \mathcal{F}_f^c(p)) \neq 0$  for each nonhyperbolic periodic point  $p$ .

#### 4. RECURRENCE PROPERTY OF THE STABLE AND UNSTABLE MANIFOLDS

Let  $\mathcal{R} = \bigcap_{n \geq 1} \mathcal{V}_n$  be the residual subset given by Proposition 3.2,  $f \in \mathcal{R}$  and  $p$  be a hyperbolic periodic point of  $f$  whose center leaf  $\mathcal{F}_f^c(p)$  is diffeomorphic to either  $S^2$  or  $\mathbb{T}^2$ . In the following we denote  $S = \mathcal{F}_f^c(p)$ , and  $g = f^k|_{\mathcal{F}_f^c(p)}$ , where  $k$  is the minimal period of  $p$ . Then we list some properties of this new map:

- (1)  $g(p) = p$ , and every periodic point of  $g : S \rightarrow S$  is elementary;
- (2)  $W_g^s(x) \pitchfork W_g^u(y)$  for any hyperbolic periodic points  $x, y$  of  $g$ ;
- (3) each nonhyperbolic periodic point has nonzero Birkhoff coefficient and is Moser stable.

**4.1. Prime-end compactification.** An important method, the *prime-end extension*, was first used by Mather [Mat81] in the study of general surface dynamics. Let  $U$  be a bounded, simply connected domain on the plane. Note that the set-theoretic boundary  $\partial U$  may be very complicate. However, there always exists a conformal map  $h : \mathbb{D} \rightarrow U$ , where  $\mathbb{D} \subset \mathbb{R}^2 = \mathbb{C}$  is the open unit disk. The prime-end compactification of  $U$  is obtained by attaching to  $U$  an ideal boundary  $\partial \mathbb{D} = S^1$  via the conformal map  $h$ . More precisely, each point  $x \in S^1$  is specified by a nested sequence of open arcs  $\gamma_n \subset U$  with  $|\gamma_n| < 1/n$  such that two endpoints of  $h^{-1}\gamma_n$  lie on both sides of  $x \in S^1$  and the sequence  $h^{-1}\gamma_n$  are nested in  $\mathbb{D}$  and converge to  $x$  (see [Mil06]). The equivalent class of this nested sequence defines a *prime point*, say  $\hat{x}$ . Denote by  $\widehat{U} \triangleq U \sqcup S^1$  the prime-end compactification of  $U$ , whose topology is uniquely determined by the extended homeomorphism  $\hat{h} : \mathbb{D} \rightarrow \widehat{U}$ , such that  $\hat{h}|_{\mathbb{D}} = h$  and  $\hat{h} : x \in S^1 \mapsto \hat{x}$ . It is important to note the relations between  $\widehat{U}$  and the set-theoretic closure  $\overline{U} = U \cup \partial U$ :

- A prime-end point  $\hat{x} \in S^1$  may cover a set of points in  $\partial U$  [Mil06, Theorem 17.7].
- A point  $x \in \partial U$  may be lifted to a set of points in the prime-ends  $S^1$  ([Mil06, Figure 37-(b)]).

See also [Mat82, §7] for various examples.

Prime-end compactifications can also be defined for any connected open subset on a closed surface  $S$ . Let  $U \subset S$  be an open connected subset on  $S$ , whose boundary consists of a finite number of connected pieces, each of these boundary pieces has more than one point. Then we can attach to  $U$  a finite number of circles, to get its prime-end compactification  $\widehat{U}$ . See also [Mat82, Xia06a].

**4.2. Prime-end extensions.** Let  $g : U \rightarrow U$  be a homeomorphism. Then there exists uniquely an extension of  $g$  to  $\widehat{U}$ , say  $\hat{g} : \widehat{U} \rightarrow \widehat{U}$ . If  $g$  is orientation-preserving, then the restriction  $\hat{g}|_{S^1}$  is an orientation-preserving circle homeomorphism. The rotation number  $\rho(\hat{g}|_{S^1})$  is called the *Carathéodory rotation number* of the map  $g$  on  $U$ , see [Mat81]. It is well known that  $\hat{g}|_{S^1}$  has periodic orbits if and only if  $\rho(\hat{g}|_{S^1})$  is rational. Moreover, we have

**Lemma 4.1.** *Let  $g$  be a symplectic diffeomorphism on a closed surface  $S$ , such that each fixed point of  $g$  is either hyperbolic, or elliptic and Moser stable. Let  $U \subset S$  be an open connected,  $g$ -invariant subset, and  $\hat{g} : \hat{U} \rightarrow \hat{U}$  be the induced prime-end extension of  $g$  on  $U$ . If  $\hat{g}$  has a fixed point on  $S^1 \subset \hat{U}$ , then  $g$  has a hyperbolic fixed point on  $\partial U$ .*

See [FL03] for a proof of the above lemma when  $S = S^2$ . Their proof actually works for any surface, see [XZ14].

**4.3. Recurrence of invariant manifolds.** Let  $p$  be a hyperbolic periodic point of  $g$ ,  $W^u(p)$  be the unstable manifold of  $p$ , which is an immersed curve passing through  $p$ . Let  $L \subset W^u(p) \setminus \{p\}$  be a branch of the unstable manifold. Without loss of generality, we assume  $g$  also fixes  $L$  (otherwise, let's consider  $g = f^{2k}$ ). Pick  $x \in L$ . Then  $I = [x, gx]$  forms a fundamental interval of  $L$ , in the sense that the intervals  $g^n(I)$ ,  $n \in \mathbb{Z}$  have mutually disjoint interiors, and the union  $\bigcup_{n \in \mathbb{Z}} g^n(I) = L$ . Then  $L$  is said to be *recurrent*, if  $L \subset \omega(L) := \limsup_{n \rightarrow \infty} g^n(I)$ . Note that this definition is independent of the choices of  $x \in L$ . There are cases when some branch of the unstable/stable manifolds is not recurrent. In particular, a branch  $L$  of an unstable manifold is said to form a *saddle connection* if  $L$  is also a branch of the stable manifold of a hyperbolic fixed point  $q$ . In the case  $q = p$ ,  $L$  is said to be a *homoclinic loop*.

**Proposition 4.1.** *Let  $g : S \rightarrow S$  be a symplectic diffeomorphism such that each periodic point of  $g$  is either hyperbolic, or elliptic and Moser stable. Then for any branch  $L$  of the invariant manifolds of any hyperbolic periodic point  $p$ , we have the following dichotomy:*

- (1) *either  $\omega(L) \supset L$ : then  $L$  is recurrent.*
- (2) *or  $\omega(L) = \{q\}$ : then  $L$  forms a saddle connection between  $p$  and  $q$ .*

The proof relies on the study of the prime-end compactification of a connected component of  $S \setminus \bar{L}$ , see [XZ14] for more details.

As a corollary, we have the following characterization of the closure of branches of stable and unstable manifolds:

**Corollary 2** ([Mat81]). *Let  $g : S \rightarrow S$  be a symplectic diffeomorphism such that each periodic point of  $g$  is either hyperbolic, or elliptic and Moser stable. If  $g$  has no saddle connection, then all four branches of the stable and unstable manifolds of any hyperbolic periodic point  $x$  of  $g$  are recurrent and have the same closure.*

## 5. HOMOCLINIC INTERSECTIONS OF HYPERBOLIC PERIODIC POINTS

Let  $\mathcal{N}^r(2) \subset \text{PH}_\omega^r(M, 2)$  be the set of maps in  $\text{PH}_\omega^r(M, 2)$  that  $r$ -normally hyperbolic (with center dimension 2) and are stably dynamically coherent. Let  $\mathcal{R}$  be the set given by Proposition 3.2, which contains residual subset of  $\mathcal{N}^r(2)$ . Let  $f \in \mathcal{R}$ , and  $p$  be a hyperbolic periodic point of  $f$  with period  $k$ . We will prove that  $W^s(p, f) \cap W^u(p, f) \setminus \{p\} \neq \emptyset$  in the case that the center leaf  $\mathcal{F}^c(p) = S^2$  or  $\mathbb{T}^2$ . In the following we denote  $S = \mathcal{F}^c(p)$  and  $g = f^k|_S$  for short. Note that it suffices to show that being a hyperbolic fixed point of the surface diffeomorphism  $g : S \rightarrow S$ ,  $W^s(p, g) \cap W^u(p, g) \setminus \{p\} \neq \emptyset$ . Then  $W^s(p, f) \cap W^u(p, f) \setminus \{p\} \supset W^s(p, g) \cap W^u(p, g) \setminus \{p\} \neq \emptyset$ , and the intersection must be transverse due to the choice of  $f \in \mathcal{R}$ .

**5.1. Spherical center leaf.** We first assume  $S$  is diffeomorphic to  $S^2$ . Our proof of the existence of homoclinic intersections in this spherical case follows the same *closing gate* approach used in [Rob73], see also [Pix82, Oli87, XZ14].

We argue by contradiction. Suppose there is no homoclinic intersection of  $p$ . Let  $L$  be a branch of the unstable manifold of  $p$ . Pick a local coordinate system  $(U, (x, y))$  around  $p$  such that  $L$  leaves  $p$  along the positive  $x$ -axis and is recurrent through the first quadrant, and the stable manifold of  $p$  moves along the  $y$ -axis.

Pick  $\epsilon$  sufficiently small, and let  $S_\epsilon = \{(x, y) \in U : 0 < x, y \leq 1, xy \leq \epsilon\}$ . Let  $q$  be the first moment on  $L$  that  $L$  intersects the set  $S_\epsilon$ . Adjusting  $\epsilon$  if necessary, we may assume  $L \cap \partial S_\epsilon$ . Let



$\Gamma^u$  be the closed curve that starts from  $p$ , first travels along  $L$  to the point  $q$ , and then slide from  $q$  to  $p$  along the closing segment  $\overline{qp}$ . Then  $\Gamma^u$  is a simple closed curve, see Fig. 1.

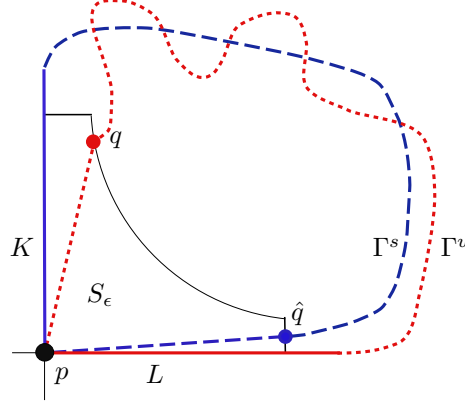


FIGURE 1. The closed curves  $\Gamma^u$  (red) and  $\Gamma^s$  (blue) obtained by closing the first intersections of  $L$  and  $K$  with  $S_\epsilon$ , respectively.

Let  $K$  be a branch of the stable manifold of  $p$  moving along the positive  $y$ -axis. Since the closure of  $K$  contains  $L$ ,  $K$  also intersects  $S_\epsilon$ . Let  $\Gamma^s$  be the corresponding simple closed curve by closing the first intersection  $\hat{q}$  of  $K$  with  $S_\epsilon$ , see also Fig. 1. Note that  $L(p, q) \cap K(p, \hat{q}) = \emptyset$ , since our hypothesis is that there is no homoclinic point. Clearly  $L(p, q) \cap \overline{p\hat{q}} = \emptyset$  and  $K(p, \hat{q}) \cap \overline{pq} = \emptyset$ , since we cut  $L$  and  $K$  at their first intersection points with  $S_\epsilon$ . Then we see that  $\Gamma^u \cap \Gamma^s = \{p\}$ , and this intersection is a topological crossing. So the algebraic intersection number  $\#(\Gamma^u, \Gamma^s) = 1$ . However, the algebraic intersection number between any two closed curves on  $S^2$  must be 0, and we arrive at a contradiction. Therefore, the hypothesis that  $p$  has no homoclinic intersection must be false. This completes the proof when  $S$  is diffeomorphic to a sphere.

**Remark 5.1.** Note that the same argument applies to general surface, as long as the algebraic intersection number of two closing curves  $C$  and  $\hat{C}$  is not 1. See Lemma 5.2 for the toric case.

**5.2. Toric center leaf.** In this subsection we assume  $S = \mathbb{T}^2$ . We will argue by contradiction. Beside the above closing gate technique, Oliveira [Oli87] took advantage of the property that the homotopy group of  $\mathbb{T}^2$  is commutative. Our proof uses the same idea of Oliveira, but much shorter. For example, it is not necessary to consider the lift of the diffeomorphism  $g$  to  $\mathbb{R}^2$ . We will show that the geometric picture of the lifts of branches is sufficient to derive a contradiction.

Assume there is no homoclinic intersection of  $p$ . We take a local coordinate system around  $p$  such that the local unstable and stable manifolds are along  $x$ -axis and  $y$ -axis, respectively. Consider canonical projection  $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$  from its universal cover with  $\pi(\mathbb{Z}^2) = p$ . Then we lift the stable and unstable manifolds  $W^{s,u}(p)$  of  $p$  to  $\mathbb{R}^2$ , and denote them by  $W^{s,u}(\mathbf{n})$  for each  $\mathbf{n} \in \mathbb{Z}^2$ . It is easy to see that none of branches of the lifted stable and unstable manifolds in  $\mathbb{R}^2$  intersect with each other, since the intersection, if exists, would induce a homoclinic intersection on  $\mathbb{T}^2$ .

**Lemma 5.1.** *Let  $L_{\mathbf{n}}$  be the lift of a branch  $L \subset W^{u,s}(p) \setminus \{p\}$  at  $\mathbf{n}$ , and  $\omega(L_{\mathbf{n}})$  be the omega limit set of  $L_{\mathbf{n}}$  in  $\mathbb{R}^2$ . Then we have:*

- (1) *if  $L_{\mathbf{n}} \subset \omega(L_0)$  for some  $\mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}$ , then  $L_{k\mathbf{n}} \subset \omega(L_0)$  for all  $k \geq 1$ .*
- (2) *if  $L_0$  is bounded, then  $L_0 \subset \omega(L_0)$ .*

*Proof.* (1) Note that if  $L_{\mathbf{n}} \subset \omega(L_0)$ , then  $\omega(L_{\mathbf{n}}) \subset \omega(L_0)$ . Therefore,  $L_{2\mathbf{n}} \subset \omega(L_{\mathbf{n}}) \subset \omega(L_0)$ . By an induction on  $k$ , we see that  $L_{(k+1)\mathbf{n}} \subset \omega(L_{k\mathbf{n}}) \subset \omega(L_0)$  for all  $k \geq 1$ . Note that  $L_0$  is unbounded.

(2) It follows from (1) that  $L_{\mathbf{n}} \cap \omega(L_0) = \emptyset$  for any  $\mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}$ . The recurrence of  $L$  on  $\mathbb{T}^2$  and the boundedness of  $L_0$  implies  $L_{\mathbf{n}} \subset \omega(L_0)$  for some  $\mathbf{n} \in \mathbb{Z}^2$ . Putting them together, we have  $L_0 \subset \omega(L_0)$ .  $\square$

Since our hypothesis is that  $p$  has no homoclinic intersection, we have the following result:

**Lemma 5.2.** *All four branches of  $W_{\pm}^{s,u}(0)$  are unbounded in  $\mathbb{R}^2$ .*

*Proof.* We argue by contradiction. Assume one of the branches of  $W_{\pm}^{s,u}(0)$ , say  $L_0$ , is bounded. Then any other branch, say  $K_0$ , is also bounded, since  $K_{\mathbf{n}} \subset \omega(L_0)$  for some  $\mathbf{n} \in \mathbb{Z}^2$ . Then it follows from Lemma 5.1 that  $L_0 \subset \omega(L_0)$  for any branch  $L_0$  of  $W_{\pm}^{s,u}$ . As in §5.1, we can find two adjacent branches of  $W_{\pm}^{s,u}(0)$  that they accumulate to themselves via the quadrant between them on  $\mathbb{R}^2$ . Then we construct on  $\mathbb{R}^2$  the two closed curves  $C$  and  $\hat{C}$  crossing each other at 0, and deduce that their intersection number is also 1. However, this contradicts the fact that the algebraic intersection number of two simple closed curves in  $\mathbb{R}^2$  must be zero. This completes the proof.  $\square$

We start with two branches on  $\mathbb{T}^2$  that accumulate to themselves via the quadrant between them, and consider their lift at  $0 \in \mathbb{R}^2$ . Without loss of generality, we assume they are  $W_+^u(0)$  and  $W_+^s(0)$ , and put them on positive  $x$ -axis and positive  $y$ -axis locally. Let  $S_{\epsilon} = \{(x, y) \in \mathbb{R}^2 : 0 < x, y \leq \epsilon, xy \leq \epsilon^2\}$  (for some small  $\epsilon$ ), and  $S_{\epsilon}(\mathbf{n}) = \mathbf{n} + S_{\epsilon}$ . We also denote  $S_{\epsilon}(\mathbb{Z}^2) = \bigcup_{\mathbb{Z}^2} S_{\epsilon}(\mathbf{n})$  for short.

Let  $S_{\epsilon}(\mathbf{n}_+^u)$  be the place where the first intersection of  $W_+^u(0) \cap S_{\epsilon}(\mathbb{Z}^2)$  happens, and  $S_{\epsilon}(\mathbf{n}_+^s)$  be the place where the first intersection of  $W_+^s(0) \cap S_{\epsilon}(\mathbb{Z}^2)$  happens.

**Lemma 5.3.** *Assume that there is no homoclinic point. Then  $\mathbb{Z}\{\mathbf{n}_+^u, \mathbf{n}_+^s\} = \mathbb{Z}^2$ .*

*Proof.* Let  $q_+^{s,u}$  be the point of the first intersection  $W_+^{s,u}(0) \cap S_{\epsilon}(\mathbb{Z}^2)$ , respectively. Let  $\gamma_+^u$  be the projection of  $\Gamma_+^u := W_+^u(0, q_+^u) * [q_+^u, \mathbf{n}_+^u]$  to  $\mathbb{T}^2$ , where  $[q_+^u, \mathbf{n}_+^u]$  is the interval connecting  $q_+^u$  and  $\mathbf{n}_+^u$ . Similarly, we define  $\Gamma_+^s$  and its projection  $\gamma_+^s$ . Non-existence of homoclinic intersection implies that the intersection number  $\#(\gamma_+^u, \gamma_+^s) = 1$ . Therefore  $\det(\mathbf{n}_+^u, \mathbf{n}_+^s) = \pm 1$  and  $\mathbb{Z}\{\mathbf{n}_+^u, \mathbf{n}_+^s\} = \mathbb{Z}^2$ .  $\square$

Now let's consider the curve  $\Gamma_1$  obtained by uniting  $\Gamma_+^u, \mathbf{n}_+^u + \Gamma_+^s, \mathbf{n}_+^s + \Gamma_+^u$  and  $\Gamma_+^s$ . This is a simply closed curve in  $\mathbb{R}^2$ , which bounds a simply connected domain, say  $Q_{\epsilon}$ . Moreover, the translations of  $Q_{\epsilon}$  by  $\mathbb{Z}\{\mathbf{n}_+^u, \mathbf{n}_+^s\} = \mathbb{Z}^2$  are disjoint and their union covers the whole plane. In other words,  $Q_{\epsilon}$  is a fundamental domain. See Fig. 2 for two illustrations of  $Q_{\epsilon}$ .

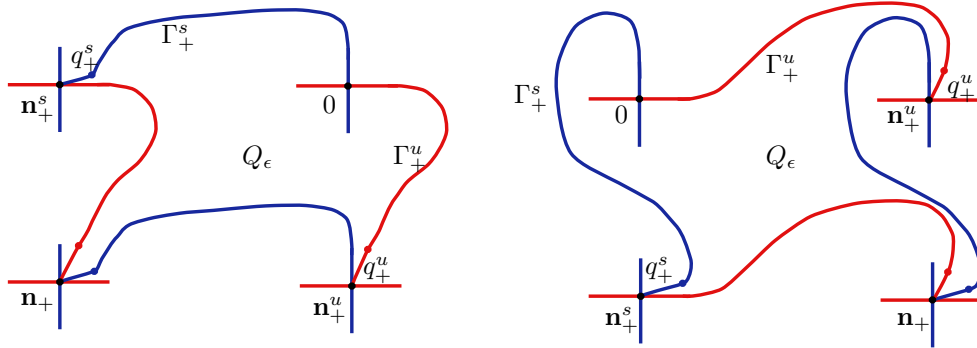


FIGURE 2. Two illustrations of the closing curves  $\Gamma_1$  and the corresponding domain  $Q_{\epsilon}$ .

Now let's describe the 4 corners of  $Q_{\epsilon}$ . It is easy to see  $Q_{\epsilon}$  contains a vertical thin wedge in  $S_{\epsilon}(\mathbf{n}_+^u)$ , a horizontal thin wedge in  $S_{\epsilon}(\mathbf{n}_+^s)$ , and an acute wedge in  $S_{\epsilon}(\mathbf{n}_+)$ , where  $\mathbf{n}_+ = \mathbf{n}_+^u + \mathbf{n}_+^s$ . The projection of the union of these three wedges covers only the first quadrant at  $p$ . Therefore,  $Q_{\epsilon}$  has to contain all three quadrants from the second to the fourth quadrant around 0. In particular, it contains two local branches:  $W_{-,loc}^u(0)$  and  $W_{-,loc}^s(0)$ .

Both branches  $W_-^u(0)$  and  $W_-^s(0)$  are unbounded, and they can't stay in  $Q_{\epsilon}$  forever. Let  $q_-^u$  and  $q_-^s$  be the points of the first intersections of  $W_-^u(0)$  and  $W_-^s(0)$  with  $\Gamma_1 = \partial Q_{\epsilon}$ , respectively. These two points must lie on the gates of  $\Gamma_1$ , since all other parts of  $\Gamma_1$  are on  $W_+^u(0)$  or  $W_+^s(0)$ . Let  $[q_-^u, \mathbf{n}_-^u]$  be the closing gate starting at  $q_-^u$ , and  $\gamma_-^u$  be the projection of  $\Gamma_-^u = W_-^u(0, q_-^u) * [q_-^u, \mathbf{n}_-^u]$  to  $\mathbb{T}^2$ . Similarly we define  $\mathbf{n}_-^s, \Gamma_-^s$  and its projection  $\gamma_-^s$ .

**Lemma 5.4.** *Assume that there is no homoclinic point. Then  $\mathbf{n}_-^u = \mathbf{n}_-^s = \mathbf{n}_+$ . In particular, the homotopy class of both  $\gamma_-^{u,s}$  is  $\mathbf{n}_+$ .*

*Proof.* First let's observe that

- $q_-^u$  lies either on the gate  $(q_+^u, \mathbf{n}_+^u)$  or on  $(\mathbf{n}_+^s + q_+^u, \mathbf{n}_+)$ , since it is on an unstable branch;
- $q_-^s$  lies either on the gate  $(q_+^s, \mathbf{n}_+^s)$  or on  $(\mathbf{n}_+^u + q_+^s, \mathbf{n}_+)$ , since it is on a stable branch.

Suppose  $q_-^u$  lies on  $(q_+^u, \mathbf{n}_+^u)$ . There are two cases when computing the intersection number of  $\gamma_-^u$  with  $\gamma_-^s$ :

- a)  $q_-^s$  lies on  $(q_+^s, \mathbf{n}_+^s)$ : then  $\gamma_-^s$  is in the class of  $\mathbf{n}_+^s$ , and  $\#(\gamma_-^u, \gamma_-^s) = |\det(\mathbf{n}_+^u, \mathbf{n}_+^s)| = 1$ ;
- b)  $q_-^s$  lies on  $(\mathbf{n}_+^u + q_+^s, \mathbf{n}_+)$ : then  $\gamma_-^s$  is in the class of  $\mathbf{n}_+$ , and  $\#(\gamma_-^u, \gamma_-^s) = |\det(\mathbf{n}_+^u, \mathbf{n}_+)| = |\det(\mathbf{n}_+^u, \mathbf{n}_+^s)| = 1$ .

On the other hand,  $\gamma_-^u \cap \gamma_-^s = \{p\}$ , since  $q_-^{u,s}$  is the first intersection of  $W_-^{u,s}(0)$  with  $\Gamma_1$ , respectively. Moreover, the intersection at  $p$  is not a topological crossing. Then the intersection number  $\#(\gamma_-^u, \gamma_-^s) = 0$ , a contradiction. Therefore,  $q_-^u$  lies on  $(\mathbf{n}_+^s + q_+^u, \mathbf{n}_+)$ . Similarly, one can prove  $q_-^s$  lies on  $(\mathbf{n}_+^u + q_+^s, \mathbf{n}_+)$ . This completes the proof.  $\square$

Next we consider another curve  $\Gamma_2$  obtained by uniting the following

- $\Gamma_+^u, \mathbf{n}_+^u + \Gamma_-^u$ , and  $\mathbf{n}_+ + \Gamma_+^u$ ;
- $\Gamma_+^s, \mathbf{n}_+^s + \Gamma_-^s$ , and  $\mathbf{n}_+ + \Gamma_+^s$ .

This is a closed curve in  $\mathbb{R}^2$ , since both  $\Gamma_-^u$  and  $\Gamma_-^s$  end at the point  $\mathbf{n}_+$ . Moreover,  $\Gamma_2$  is a simply closed curve since we are working under the hypothesis that there is no homoclinic point. Let  $R_\epsilon$  be the simply connected domain in  $\mathbb{R}^2$  bounded by  $\Gamma_2$ . Note that  $R_\epsilon$  also contains the two local branches:  $W_{-,loc}^u(0)$  and  $W_{-,loc}^s(0)$ . Again none of the branches  $W_-^u(0)$  and  $W_-^s(0)$  can't stay in  $Q_\epsilon$  forever since they are unbounded. Let  $x_-^u$  and  $x_-^s$  be the point of the first intersection of  $W_-^u(0)$  and  $W_-^s(0)$  with  $\Gamma_2 = \partial R_\epsilon$ , respectively. Note that each of the six components used to define  $\Gamma_2$  contains a closing segment. Following a similar reasoning for determining the locations of  $q_-^{u,s}$ , we have

- (1)  $x_-^u$  lies on either  $(q_+^u, \mathbf{n}_+^u)$ , or  $(\mathbf{n}_+^u + q_-^u, \mathbf{n}_+^u + \mathbf{n}_+)$ , or  $(\mathbf{n}_+ + q_+^u, \mathbf{n}_+ + \mathbf{n}_+^u)$ ;
- (2)  $x_-^s$  lies on either  $(q_+^s, \mathbf{n}_+^s)$ , or  $(\mathbf{n}_+^s + q_-^s, \mathbf{n}_+^s + \mathbf{n}_+)$ , or  $(\mathbf{n}_+ + q_+^s, \mathbf{n}_+ + \mathbf{n}_+^s)$ .

Now consider the projection  $\gamma_2^u$  of  $\Gamma_2^u := W_-^u(0, x_-^u) * [x_-^u, \mathbf{n}(x_-^u)]$ , where  $\mathbf{n}(x_-^u) \in \{\mathbf{n}_+^u, \mathbf{n}_+^u + \mathbf{n}_+\}$  (depending on the location of  $x_-^u$ ). Similarly we define  $\Gamma_2^s := W_-^s(0, x_-^s) * [x_-^s, \mathbf{n}(x_-^s)]$  and its projection  $\gamma_2^s$ , where  $\mathbf{n}(x_-^s) \in \{\mathbf{n}_+^s, \mathbf{n}_+^s + \mathbf{n}_+\}$  (depending on the location of  $x_-^s$ ). In any combination of the possible locations of the two points  $x_-^u$  and  $x_-^s$ , we always have

$$\#(\gamma_2^u, \gamma_2^s) = |\det(\mathbf{n}(x_-^u), \mathbf{n}(x_-^s))| = |\det(\mathbf{n}_+^u, \mathbf{n}_+^s)| = 1. \quad (5.1)$$

On the other hand,  $\gamma_2^u \cap \gamma_2^s = \{p\}$ , since  $x_-^{u,s}$  is the first intersection of  $W_-^{u,s}(0)$  with  $\Gamma_2$ , respectively. Moreover, the intersection at  $p$  is not a topological crossing. Then the intersection number  $\#(\gamma_2^u, \gamma_2^s) = 0$ , which contradicts Eq. (5.1). This completes the proof.

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#### REFERENCES

- [Ano67] D. V. Anosov. *Geodesic flows on closed Riemann manifolds with negative curvature*. Trudy Mat. Inst. Steklov. **90**, 1967.
- [Bir35] G. Birkhoff. Nouvelles recherches sur les systemes dynamiques. Memoriae Pont. Acad. Sci. Novi Lyncaei **1** (1935), 85–216; Collected Math. Papers **II** (1968), 530–659.
- [BW08] K. Burns, A. Wilkinson. *Dynamical coherence and center bunching*. Discrete Contin. Dynam. Syst. **22** (2008), 89–100.
- [FL03] J. Franks, P. Le Calvez. *Regions of instability for non-twist maps*. Ergod. Th. Dynam. Syst. **23** (2003), 111–141.
- [HPS] M. Hirsch, C. Pugh, M. Shub. *Invariant Manifolds*. Lect. Notes Math. **583**, Springer-Verlag, 1977.

- [Mat81] J. Mather. *Invariant subsets of area-preserving homeomorphisms of surfaces*. Adv. Math. Suppl. Stud. **7B** (1981) 531–62.
- [Mat82] J. Mather. *Topological proofs of some purely topological consequences of Caratheodory’s theory of prime ends*. In: Selected Studies, Eds. Th. M. Rassias, G. M. Rassias, 1982, 225–255.
- [Mel63] V. K. Melnikov. *On the stability of the center for time periodic perturbations*. Trans. Moscow Math. Soc. **12** (1963), 1–57.
- [Mil06] J. Milnor. *Dynamics in one complex variable*. 3rd edition. Annals Math. Studies **160**, Princeton University Press, Princeton, NJ, 2006.
- [Mos73] J. Moser. *Stable and Random Motions in Dynamical Systems*. Annals Math. Studies **77**, Princeton, NJ, 1973.
- [Oli87] F. Oliveira. *On the generic existence of homoclinic points*. Ergod. Th. Dynam. Syst. **7** (1987), 567–595.
- [Oli00] F. Oliveira. *On the  $C^\infty$  genericity of homoclinic orbits*. Nonlinearity **13** (2000), 653–662.
- [Pes77] Ya. Pesin. *Lyapunov characteristic exponents and smooth ergodic theory*. Russian Math. Surv. **32** (1977), 55–114.
- [Pix82] D. Pixton. *Planar homoclinic points*. J. Differential Equations **44** (1982), 365–382.
- [Poin] H. Poincare. *Les methodes nouvelles de la mecanique celeste*. (French) [New methods of celestial mechanics.] Gauthier-Villars, Paris, vol. 1 in 1892; vol 2 in 1893; vol. 3 in 1899.
- [Pug67a] C. Pugh. *The closing lemma*. Amer. J. Math. **89** (1967), 956–1009.
- [Pug67b] C. Pugh. *An improved closing lemma and a general density theorem*. Amer. J. Math. **89** (1967), 1010–1021.
- [Pug11] C. Pugh. *The closing lemma in retrospect*. Dynamics, games and science. I, 721–741, Springer, Heidelberg, 2011.
- [PuRo83] C. Pugh, C. Robinson. *The  $C^1$  closing lemma, including Hamiltonians*. Ergod. Theor. Dyn. Sys. **3** (1983), 261–313.
- [Rob70] C. Robinson. *Generic properties of conservative systems*. Amer. J. Math. **92** (1970) 562–603.
- [Rob73] C. Robinson. *Closing stable and unstable manifolds in the two-sphere*. Proc. Am. Math. Soc. **41** (1973), 299–303.
- [SX06] R. Saghin, Z. Xia. *Partial hyperbolicity or dense elliptic periodic points for  $C^1$ -generic symplectic diffeomorphisms*. Trans. Amer. Math. Soc. **358** (2006), 5119–5138.
- [Sma65] S. Smale. *Diffeomorphisms with many periodic points*. Differential and combinatorial topology, Princeton Univ. Press (1965), 63–80.
- [Tak72] F. Takens. *Homoclinic points in conservative systems*. Invent. Math. **18** (1972), 267–292.
- [Xia96] Z. Xia. *Homoclinic points in symplectic and volume-preserving diffeomorphisms*. Comm. Math. Phys. **177** (1996), 435–449.
- [Xia06a] Z. Xia. *Area-preserving surface diffeomorphisms*. Comm. Math. Phys. **263** (2006), 723–735.
- [Xia06b] Z. Xia. *Homoclinic points for area-preserving surface diffeomorphisms*. [arxiv.org/abs/math/0606291](https://arxiv.org/abs/math/0606291).
- [XZ06] Z. Xia, H. Zhang. *A  $C^r$  closing lemma for a class of symplectic diffeomorphisms*. Nonlinearity **19** (2006) 511–516.
- [XZ14] Z. Xia, P. Zhang. *Homoclinic points for convex billiards*. Nonlinearity **27** (2014), 1181–1192.
- [Zh15] P. Zhang. *Convex billiards on convex spheres*. [arxiv.org/abs/1505.01418](https://arxiv.org/abs/1505.01418).

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